11-3 Geometric Sequences

Main Ideas

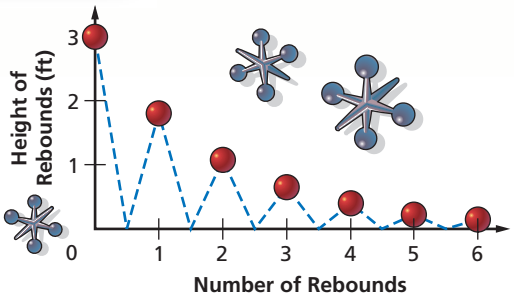
- Use geometric sequences.
- Find geometric means.

New Vocabulary

geometric sequence
common ratio
geometric means

GET READY for the Lesson

When you drop a ball, it never rebounds to the height from which you dropped it. Suppose a ball is dropped from a height of three feet, and each time it falls, it rebounds to 60% of the height from which it fell. The heights of the ball's rebounds form a sequence.



Geometric Sequences The height of the first rebound of the ball is 3(0.6) or 1.8 feet. The height of the second rebound is 1.8(0.6) or 1.08 feet. The height of the third rebound is 1.08(0.6) or 0.648 feet. The sequence of heights is an example of a **geometric sequence**. A geometric sequence is a sequence in which each term after the first is found by multiplying the previous term by a nonzero constant r called the **common ratio**.

As with an arithmetic sequence, you can label the terms of a geometric sequence as a_1, a_2, a_3, and so on, $a_1 \neq 0$. The nth term is a_n and the previous term is a_{n-1}. So, $a_n = r(a_{n-1})$. Thus, $r = \frac{a_n}{a_{n-1}}$. That is, the common ratio can be found by dividing any term by its previous term.

STANDARDIZED TEST EXAMPLE Find the Next Term

1 What is the missing term in the geometric sequence: 8, 20, 50, 125, ____ ?

A 75 **B** 200 **C** 250 **D** 312.5

Test-Taking Tip

Since the terms of this sequence are increasing, the missing term must be greater than 125. You can immediately eliminate 75 as a possible answer.

Read the Test Item

Since $\frac{20}{8} = 2.5$, $\frac{50}{20} = 2.5$, and $\frac{125}{50} = 2.5$, the common ratio is 2.5.

Solve the Test Item

To find the missing term, multiply the last given term by 2.5: 125(2.5) = 312.5. The answer is D.

CHECK Your Progress

1. What is the missing term in the geometric sequence: −120, 60, −30, 15, _____ ?

F −7.5 **G** 0 **H** 7.5 **J** 10

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You have seen that each term of a geometric sequence after the first term can be expressed in terms of r and its previous term. It is also possible to develop a formula that expresses each term of a geometric sequence in terms of r and the first term a_1. Study the patterns in the table for the sequence 2, 6, 18, 54, … .

Sequence	**numbers**	2	6	18	54	…	
	symbols	a_1	a_2	a_3	a_4	…	a_n
Expressed in Terms of *r* and the Previous Term	**numbers**	2	2(3)	6(3)	18(3)	…	
	symbols	a_1	$a_1 \cdot r$	$a_2 \cdot r$	$a_3 \cdot r$	…	$a_{n-1} \cdot r$
Expressed in Terms of *r* and the First Term	**numbers**	2	2(3)	2(9)	2(27)	…	
		$2(3^0)$	$2(3^1)$	$2(3^2)$	$2(3^3)$	…	
	symbols	$a_1 \cdot r^0$	$a_1 \cdot r^1$	$a_1 \cdot r^2$	$a_1 \cdot r^3$	…	$a_1 \cdot r^{n-1}$

The three entries in the last column all describe the nth term of a geometric sequence. This leads to the following formula.

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KEY CONCEPT ***n*th Term of a Geometric Sequence**

The nth term a_n of a geometric sequence with first term a_1 and common ratio r is given by the following formula, where n is any positive integer.

$$a_n = a_1 \cdot r^{n-1}$$

Study Tip

Finding a Term

For small values of r and n, it may be easier to multiply by r successively to find a given term than to use the formula.

EXAMPLE **Find a Term Given the First Term and the Ratio**

2 **Find the eighth term of a geometric sequence for which $a_1 = -3$ and $r = -2$.**

$a_n = a_1 \cdot r^{n-1}$ Formula for nth term

$a_8 = (-3) \cdot (-2)^{8-1}$ $n = 8, a_1 = -3, r = -2$

$a_8 = (-3) \cdot (-128)$ $(-2)^7 = -128$

$a_8 = 384$ Multiply.

CHECK Your Progress

2. Find the sixth term of a geometric sequence for which $a_1 = -\frac{1}{9}$ and $r = 3$.

EXAMPLE **Write an Equation for the *n*th Term**

3 **Write an equation for the nth term of the geometric sequence 3, 12, 48, 192, … .**

$a_n = a_1 \cdot r^{n-1}$ Formula for nth term

$a_n = 3 \cdot 4^{n-1}$ $a_1 = 3, r = 4$

CHECK Your Progress

3. Write an equation for the nth term of the geometric sequence 18, −3, $\frac{1}{2}$, $-\frac{1}{12}$, … .

You can also use the formula for the nth term if you know the common ratio and one term of a geometric sequence, but not the first term.

EXAMPLE 4 Find a Term Given One Term and the Ratio

Find the tenth term of a geometric sequence for which $a_4 = 108$ and $r = 3$.

Step 1 Find the value of a_1.

$a_n = a_1 \cdot r^{n-1}$	Formula for nth term
$a_4 = a_1 \cdot 3^{4-1}$	$n = 4, r = 3$
$108 = 27a_1$	$a_4 = 108$
$4 = a_1$	Divide each side by 27.

Step 2 Find a_{10}.

$a_n = a_1 \cdot r^{n-1}$	Formula for nth term
$a_{10} = 4 \cdot 3^{10-1}$	$n = 10, a_1 = 4, r = 3$
$a_{10} = 78{,}732$	Use a calculator.

The tenth term is 78,732.

CHECK Your Progress

4. Find the eighth term of a geometric sequence for which $a_3 = 16$ and $r = 4$.

Geometric Means In Lesson 11-1, you learned that missing terms between two nonsuccessive terms in an arithmetic sequence are called *arithmetic means*. Similarly, the missing terms(s) between two nonsuccessive terms of a geometric sequence are called **geometric means**. For example, 6, 18, and 54 are three geometric means between 2 and 162 in the sequence 2, 6, 18, 54, 162, … . You can use the common ratio to find the geometric means in a sequence.

EXAMPLE 5 Find Geometric Means

Study Tip

Alternate Method

You may prefer this method. The three means will be $2.25r$, $2.25r^2$, and $2.25r^3$. Then the common ratio is $r = \frac{576}{2.25r^3}$ or $r^4 = \frac{576}{2.25}$. Thus, $r = 4$.

Find three geometric means between 2.25 and 576.

Use the nth term formula to find the value of r. In the sequence 2.25, $\frac{?}{}$, $\frac{?}{}$, $\frac{?}{}$, 576, a_1 is 2.25 and a_5 is 576.

$a_n = a_1 \cdot r^{n-1}$	Formula for nth term
$a_5 = 2.25 \cdot r^{5-1}$	$n = 5, a_1 = 2.25$
$576 = 2.25r^4$	$a_5 = 576$
$256 = r^4$	Divide each side by 2.25.
$\pm 4 = r$	Take the fourth root of each side.

There are two possible common ratios, so there are two possible sets of geometric means. Use each value of r to find three geometric means.

$r = 4$	$r = -4$
$a_2 = 2.25(4)$ or 9	$a_2 = 2.25(-4)$ or -9
$a_3 = 9(4)$ or 36	$a_3 = -9(-4)$ or 36
$a_4 = 36(4)$ or 144	$a_4 = 36(-4)$ or -144

The geometric means are 9, 36, and 144, or -9, 36, and -144.

CHECK Your Progress

5. Find two geometric means between 4 and 13.5.

CHECK Your Understanding

Example 1 (p. 636)

1. Find the next two terms of the geometric sequence 20, 30, 45, … .
2. Find the first five terms of the geometric sequence for which $a_1 = -2$ and $r = 3$.

3. **STANDARDIZED TEST PRACTICE** What is the missing term in the geometric sequence: $-\frac{1}{4}, \frac{1}{2}, -1, 2, \_\_\_$?

   A -4 **B** $-3\frac{1}{2}$ **C** $3\frac{1}{2}$ **D** 4

Example 2 (p. 637)

4. Find a_9 for the geometric sequence 60, 30, 15, … .
5. Find a_8 for the geometric sequence $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \ldots$.

Find the indicated term of each geometric sequence.

6. $a_1 = 7, r = 2, n = 4$
7. $a_1 = 3, r = \frac{1}{3}, n = 5$

Example 3 (p. 637)

8. Write an equation for the nth term of the geometric sequence 4, 8, 16, … .
9. Write an equation for the nth term of the geometric sequence 15, 5, $\frac{5}{3}$, … .

Example 4 (p. 638)

Find the indicated term of each geometric sequence.

10. $a_3 = 24, r = \frac{1}{2}, n = 7$
11. $a_3 = 32, r = -0.5, n = 6$

Example 5 (p. 638)

12. Find two geometric means between 1 and 27.
13. Find two geometric means between 2 and 54.

Exercises

HOMEWORK HELP

For Exercises	See Examples
14–19	1
20–27	2
28, 29	3
30–33	4
34–37	5

Find the next two terms of each geometric sequence.

14. 405, 135, 45, …
15. 81, 108, 144, …
16. 16, 24, 36, …
17. 162, 108, 72, …

Find the first five terms of each geometric sequence described.

18. $a_1 = 2, r = -3$
19. $a_1 = 1, r = 4$
20. Find a_7 if $a_1 = 12$ and $r = \frac{1}{2}$.
21. Find a_6 if $a_1 = \frac{1}{3}$ and $r = 6$.

22. **INTEREST** An investment pays interest so that each year the value of the investment increases by 10%. How much is an initial investment of $1000 worth after 5 years?

23. **SALARIES** Geraldo's current salary is $40,000 per year. His annual pay raise is always a percent of his salary at the time. What would his salary be if he got four consecutive 4% increases?

Find the indicated term of each geometric sequence.

24. $a_1 = \frac{1}{3}, r = 3, n = 8$
25. $a_1 = \frac{1}{64}, r = 4, n = 9$
26. a_9 for $a_1 = \frac{1}{5}, 1, 5, \ldots$
27. a_7 for $\frac{1}{32}, \frac{1}{16}, \frac{1}{8}, \ldots$
28. $a_4 = 16, r = 0.5, n = 8$
29. $a_6 = 3, r = 2, n = 12$

Real-World Link

The largest ever ice construction was an ice palace built for a carnival in St. Paul, Minnesota, in 1992. It contained 10.8 million pounds of ice.

Source: *The Guinness Book of Records*

Write an equation for the nth term of each geometric sequence.

30. 36, 12, 4, …

31. 64, 16, 4, …

32. −2, 10, −50, …

33. 4, −12, 36, …

Find the geometric means in each sequence.

34. 9, ?, ?, ?, 144

35. 4, ?, ?, ?, 324

36. 32, ?, ?, ?, ?, 1

37. 3, ?, ?, ?, ?, 96

Find the next two terms of each geometric sequence.

38. $\frac{5}{2}, \frac{5}{3}, \frac{10}{9}, \ldots$

39. $\frac{1}{3}, \frac{5}{6}, \frac{25}{12}, \ldots$

40. 1.25, −1.5, 1.8, …

41. 1.4, −3.5, 8.75, …

Find the first five terms of each geometric sequence described.

42. $a_1 = 243, r = \frac{1}{3}$

43. $a_1 = 576, r = -\frac{1}{2}$

44. ART A one-ton ice sculpture is melting so that it loses one-eighth of its weight per hour. How much of the sculpture will be left after five hours? Write your answer in pounds.

MEDICINE For Exercises 45 and 46, use the following information.

Iodine-131 is a radioactive element used to study the thyroid gland.

45. RESEARCH Use the Internet or other resource to find the *half-life* of Iodine-131, rounded to the nearest day. This is the amount of time it takes for half of a sample of Iodine-131 to decay into another element.

46. How much of an 80-milligram sample of Iodine-131 would be left after 32 days?

EXTRA PRACTICE
See pages 914, 936.
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Find the indicated term of each geometric sequence.

47. $a_1 = 16{,}807, r = \frac{3}{7}, n = 6$

48. $a_1 = 4096, r = \frac{1}{4}, n = 8$

49. a_8 for 4, −12, 36, …

50. a_6 for 540, 90, 15, …

51. $a_4 = 50, r = 2, n = 8$

52. $a_4 = 1, r = 3, n = 10$

H.O.T. Problems

53. OPEN ENDED Write a geometric sequence with a common ratio of $\frac{2}{3}$.

54. FIND THE ERROR Marika and Lori are finding the seventh term of the geometric sequence 9, 3, 1, … . Who is correct? Explain your reasoning.

Marika

$r = \frac{3}{9}$ or $\frac{1}{3}$

$a_7 = 9\left(\frac{1}{3}\right)^{7-1}$

$= \frac{1}{81}$

Lori

$r = \frac{9}{3}$ or 3

$a_7 = 9 \cdot 3^{7-1}$

$= 6561$

55. Which One Doesn't Belong? Identify the sequence that does not belong with the other three. Explain your reasoning.

1, 4, 16, …

3, 9, 27, …

9, 16, 25, …

$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

CHALLENGE **Determine whether each statement is *true* or *false*. If true, explain. If false, provide a counterexample.**

56. Every sequence is either arithmetic or geometric.

57. There is no sequence that is both arithmetic and geometric.

58. ***Writing in Math*** Use the information on pages 636 and 637 to explain the relationship between n and a_n in the formula for the nth term of a geometric sequence. If n is the independent variable and a_n is the dependent variable, what kind of equation relates n and a_n? Explain what r represents in the context of the relationship.

STANDARDIZED TEST PRACTICE

59. **ACT/SAT** The first four terms of a geometric sequence are shown in the table. What is the tenth term in the sequence?

a_1	144
a_2	72
a_3	36
a_4	18

A 0

B $\frac{9}{64}$

C $\frac{9}{32}$

D $\frac{9}{16}$

60. **REVIEW** The table shows the cost of jelly beans depending on the amount purchased. Which conclusion can be made based on the table?

Cost of Jelly Beans	
Number of Pounds	Cost
5	\$14.95
20	\$57.80
50	\$139.50
100	\$269.00

F The cost of 10 pounds of jelly beans would be more than \$30.

G The cost of 200 pounds of jelly beans would be less than \$540.

H The cost of jelly beans is always more than \$2.70 per pound.

J The cost of jelly beans is always less than \$2.97 per pound.

Spiral Review

Find S_n for each arithmetic series described. (Lesson 11-2)

61. $a_1 = 11, a_n = 44, n = 23$

62. $a_1 = -5, d = 3, n = 14$

Find the arithmetic means in each sequence. (Lesson 11-1)

63. $15, \_\_?\_\_, \_\_?\_\_, 27$

64. $-8, \_\_?\_\_, \_\_?\_\_, \_\_?\_\_, -24$

65. **GEOMETRY** Find the perimeter of a triangle with vertices at $(2, 4)$, $(-1, 3)$ and $(1, -3)$. (Lesson 10-1)

GET READY for the Next Lesson

PREREQUISITE SKILL **Evaluate each expression.** (Lesson 1-1)

66. $\frac{1 - 2^7}{1 - 2}$

67. $\frac{1 - \left(\frac{1}{2}\right)^6}{1 - \frac{1}{2}}$

68. $\frac{1 - \left(-\frac{1}{3}\right)^5}{1 - \left(-\frac{1}{3}\right)}$